Assignment 5

Hand in: no. 14, Supplementary Exercises no. 2.
Deadline: Feb 23, 2018.

Section 7.1: *no. 1, 2, 8, 11, 14, 15.* (In 14(d) you apply Theorem 2.6 instead of Example 7.1.4.)

Supplementary Exercises

Use the knowledge in Section 1, Notes 2.

1. Let f be a continuous function on (a, b) satisfying

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(f(x)+f(y)\Big), \quad \forall x, y \in (a,b).$$

   Show that f is convex. Suggestion: Show

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{f(x_1)+\cdots+f(x_n)}{n} ,$$

   for $n = 2^m$.

2. Let f be differentiable on $[a, b]$. Show that it is convex if and only if

$$f(y) - f(x) \ge f'(x)(y-x), \qquad \forall x, y \in [a,b].$$

   What is the geometric meaning of this inequality?

3. Establish the following two inequalities

   (a)

$$\sin x + \sin y + \sin z \le \frac{3\sqrt{3}}{2} .$$

   (b)

$$\sin x \ \sin y \ \sin z \le \frac{3\sqrt{3}}{8} .$$

   (c)

$$\frac{1}{3}\left(\frac{1}{\sin x} + \frac{1}{\sin y} + \frac{1}{\sin z}\right) \ge \frac{2}{\sqrt{3}} .$$

   Here x, y, z are the three interior angles of a triangle.

4. Establish the inequality

$$a^a b^b c^c \ge \left(\frac{a+b+c}{3}\right)^{a+b+c} , \quad a, b, c > 0 .$$

   Hint: Use of one the functions in (1).

5. Let P be the partition $\{-1, -\frac{1}{2}, 0, \frac{1}{3}, 1\}$ of $[-1, 1]$. Define $f : [-1, 1]$ by

$$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0], \\ -x + 1 & \text{if } x \in (0, 1]. \end{cases}$$

   (a) Find the Darboux upper and lower sums for f. Explain why the Darboux upper sum is not a Riemann sum.

   (b) Use the integrability criterion to show that f is integrable and find its integral.

6. Prove Cauchy criterion for integrability: f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two tagged partitions $\dot{P}, \dot{Q}$ with length less than δ,

$$|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon,$$

holds. (This criterion is proved in the text; pretend that it is not there.)